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Symmetry Results for Semilinear Elliptic Systems in the Whole Space

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1. INTRODUCTION

In this paper we study the radial symmetry of classical solutions of elliptic systems of the following type:

$$\begin{cases} \Delta u_i + f_i(r, u_1, ..., u_n) = 0 & \text{in} \quad \mathbb{R}^N, \ i = 1, ..., n, \\ u_i > 0 & \text{in} \quad \mathbb{R}^N, \\ u_i(x) \to 0 & \text{as} \quad r = |x| \to \infty, \end{cases} \tag{1}$$

where $n \geqslant 1$, $N \geqslant 2$ are arbitrary integers.

In the case of a bounded domain, related results for autonomous systems were established by Troy [17] (see also de Figueiredo [4], Shaker [16]). If one assumes a priori asymptotic expansions of the solutions, a symmetry result in $\mathbb{R}^N$ in the spirit of Gidas, Ni and Nirenberg [11] was also proved by Shaker. We further remark that the case of a single equation has been extensively studied since the work of Gidas, Ni and Nirenberg (see for instance C. Li [12], Y. Li and W.-M. Ni [13]).

In a recent paper D. G. de Figueiredo and J. Yang [8] studied the symmetry of positive solutions of systems of two equations under some restrictive hypotheses on the nonlinearities (see Section 2.1).

Using variational methods, de Figueiredo and Yang also proved existence and decay at infinity of positive solutions of such systems. More

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general results about existence and decay can be found in [15], as well as an application of our symmetry result to the existence of a ground state of the system.

We next list our assumptions on (1). Let us note $u=(u_1, ..., u_n) \in \mathbb{R}^n_+ = (0, \infty)^n$ and

$$A(r, u^1, ..., u^n) = \left(\frac{\partial f_i}{\partial u_j} (r, u^i) \right)_{1 \leqslant i,\, j \leqslant n}$$

for $r \geqslant 0$ and $u^i \in \mathbb{R}^n_+$, $1 \leqslant i \leqslant n$. We suppose that $f_i \in C^1([0, \infty) \times \mathbb{R}^n_+, \mathbb{R})$ for $i=1, ..., n$ and

(H1) $(\partial f_i / \partial r)(r, u) \leqslant 0$ for all $(r, u) \in \mathbb{R}^{n+1}_+$ and $i=1, ..., n$;

(H2) the system is *cooperative* (or quasimonotone), that is,

$$\frac{\partial f_i}{\partial u_j} (r, u) \geqslant 0$$

for all $(r, u) \in \mathbb{R}^{n+1}_+$ and all $i, j \in \{1, ..., n\}$, $i \neq j$;

(H3) there exist constants $\varepsilon > 0$ and $R_1 > 0$ such that the system is *fully coupled* in the set

$$\mathcal{O} = \{(r, u) \mid r > R_1,\ u \in \mathbb{R}^n_+,\ |u| < \varepsilon\},$$

that is, for any $I, J \subset \{1, ..., n\}$, $I \cap J = \varnothing$, $I \cup J = \{1, ..., n\}$ there exist $i_0 \in I$ and $j_0 \in J$ such that

$$\frac{\partial f_{j_0}}{\partial u_{i_0}} (r, u) > 0$$

for all $(r, u) \in \mathcal{O}$;

(H4) all n-principal minors of $-A(r, u^1, ..., u^n)$ have nonnegative determinants, for all $(r, u^i) \in \mathcal{O}$, $1 \leqslant i \leqslant n$. We recall that the n-principal minors of a matrix $(m_{ij})_{1 \leqslant i,\, j \leqslant n}$ are the submatrices $(m_{ij})_{1 \leqslant i,\, j \leqslant k}$ with $1 \leqslant k \leqslant n$.

Assumption (H2) is widely used for elliptic systems. In particular, Troy and Shaker proved their results under (H2). Condition (H3) means that the system cannot be reduced to two independent systems. It is this fact that forces all functions u_i to be radially symmetric with respect to the same origin. Finally, (H4) is the natural generalisation of the hypothesis at infinity, used for single equations. Actually, in the scalar case

$$\Delta u + f(r, u) = 0,$$

(H2)–(H4) reduce to $(\partial f/\partial u)(r, u) \leqslant 0$ for small u and large r which is exactly the assumption considered by Y. Li and W.-M. Ni in [13] (see also C. Li [12]).

Note that the functions f_i are not assumed to be defined on points which have a zero coordinate.

Here is our main result.

THEOREM 1. *Suppose* $f_1, ..., f_n$ *satisfy* (H1)–(H4), *and* $u = (u_1, ..., u_n)$ *is a classical solution of* (1). *Then there exists a point* $x_0 \in \mathbb{R}^N$ *such that the functions* u_i *are radially symmetric with respect to the origin* x_0, *that is* $u_i(x) = u_i(|x - x_0|)$, $i = 1, ..., n$. *Moreover,*

$$\frac{du_i}{dr} < 0 \qquad \textit{for all} \quad r = |x - x_0| > 0.$$

Section 2 is devoted to the proof of Theorem 1. We start by giving the proof in a simpler setting of two autonomous equations, where the main ideas are made more explicit. In this case we are able to give a full generalisation of the hypothesis (H3). We even state a theorem that does not include it. Finally, in Section 3, we discuss our assumptions and give simple examples of nonexistence of positive solutions when one of them is not satisfied.

2. PROOF OF THE MAIN THEOREM

2.1. *The Case of Two Equations*

In this section we prove the symmetry result for classical solutions of the system

$$\begin{cases} \Delta u + g(u, v) = 0 & \text{in} \quad \mathbb{R}^N \\ \Delta v + f(u, v) = 0 & \text{in} \quad \mathbb{R}^N \\ u, v > 0 & \text{in} \quad \mathbb{R}^N \\ u(x), v(x) \to 0 & \text{as} \quad |x| \to \infty, \end{cases} \tag{2}$$

with $f, g \in C^1([0, \infty) \times [0, \infty), \mathbb{R})$. We suppose that

(i) $(\partial g/\partial v)(u, v)$ and $(\partial f/\partial u)(u, v)$ are non-negative for all $(u, v) \in [0, \infty) \times [0, \infty)$;

(ii) $(\partial g/\partial u)(0, 0) < 0$ and $(\partial f/\partial v)(0, 0) < 0$;

(iii) $\det A > 0$, where

$$A = \begin{pmatrix} \dfrac{\partial g}{\partial u} & \dfrac{\partial g}{\partial v} \\ \dfrac{\partial f}{\partial u} & \dfrac{\partial f}{\partial v} \end{pmatrix} (0, 0).$$

In order to avoid some technicalities, here we have strengthened our hypotheses from Section 1. Of course, using the method of Section 2.2 all results in Section 2.1 can be shown to hold under (H1)–(H4). We note that (ii) and (iii) are exactly the conditions under which the linearized system at zero satisfies the maximum principle (see [6] and [10]).

In [8] de Figueiredo and Yang consider the case

$$g(u, v) = -u + g_1(v), \qquad f(u, v) = -v + f_1(u)$$

where $f_1(0) = g_1(0) = f'_1(0) = g'_1(0) = 0$, f_1 and g_1 are positive and convex in $\mathbb{R}_+$, and have power-like growth at zero and infinity. Note that in this case $-A$ is the identity matrix. The result of de Figueiredo and Yang concerns only exponentially decreasing solutions of (2). None of these features is present in our work.

We have the following result.

THEOREM 2. *Assume* (i), (ii) *and* (iii) *hold. Then there exist points* $x_0, x_1 \in \mathbb{R}^N$ *such that* $u(x) = u(|x - x_0|)$ *and* $v(x) = v(|x - x_1|)$. *Moreover,*

$$\frac{du}{dr_1} < 0 \qquad \textit{and} \qquad \frac{dv}{dr_2} < 0$$

for all $r_1 = |x - x_0| > 0$ *and* $r_2 = |x - x_1| > 0$.

We see from Theorem 3 that if $x_0 \neq x_1$ then u changes its values on sets where v is constant and vice versa. Therefore, if $x_0 \neq x_1$, and both functions u and v are effectively present in one of the equations in (2), then this equation cannot be satisfied. We conclude that if v (resp. u) appears in a non-zero term in the first (resp. second) equation in (2) then necessarily the solutions are symmetric with respect to the same origin.

Sufficient conditions for $x_0 = x_1$ in Theorem 3, which do not depend on the particular choice of the solutions, are for example:

(iv)′ either $(\partial g/\partial v)$ or $(\partial f/\partial u)$ is positive in a neighbourhood of $(0, 0)$, except possibly on $\{u = 0\} \cup \{v = 0\}$;

(iv)″ either $(\partial g/\partial v)$ or $(\partial f/\partial u)$ does not depend on one of its variables and is not identically zero in every neighbourhood of $(0, 0)$.

Proof of Theorem 2. In order to prove symmetry of solutions we apply the "moving planes" method. For all $\lambda \in \mathbb{R}$ we define the hyperplane $T_\lambda = \{x \in \mathbb{R}^N \mid x_1 = \lambda\}$ and put $\Sigma_\lambda = \{x \in \mathbb{R}^N \mid x_1 > \lambda\}$. Our goal is to show that the solutions of (2) are symmetric with respect to T_λ, for some $\lambda \in \mathbb{R}$. Then we can finish the proof, as explained in the beginning of Section 2.2.

Let u and v be solutions of (2). For any point $x \in \Sigma_\lambda$ we denote with x^λ its reflexion with respect to T_λ and introduce the functions $u_\lambda(x) = u(x^\lambda)$, $v_\lambda(x) = v(x^\lambda)$, $U_\lambda(x) = u_\lambda(x) - u(x)$ and $V_\lambda(x) = v_\lambda(x) - v(x)$, all of them defined in Σ_λ. The change of variables $x \to x^\lambda$ leaves the equations in (2) unchanged so we can substract them from the corresponding ones for u_λ and v_λ, to obtain in Σ_λ

$$\Delta U_\lambda + g(u_\lambda(x), v_\lambda(x)) - g(u(x), v(x)) = 0$$

$$\Delta V_\lambda + f(u_\lambda(x), v_\lambda(x)) - f(u(x), v(x)) = 0,$$

and consequently, by Taylor's expansion,

$$\Delta U_\lambda + \frac{\partial g}{\partial u}(\xi_1(x, \lambda), v(x))\, U_\lambda + \frac{\partial g}{\partial v}(u_\lambda(x), \eta_1(x, \lambda))\, V_\lambda = 0 \tag{3}$$

$$\Delta V_\lambda + \frac{\partial f}{\partial u}(\xi_2(x, \lambda), v_\lambda(x))\, U_\lambda + \frac{\partial f}{\partial v}(u(x), \eta_2(x, \lambda))\, V_\lambda = 0, \tag{4}$$

where

$$\xi_i(x, \lambda) \in (\min\{u(x), u_\lambda(x)\}, \max\{u(x), u_\lambda(x)\}),$$

$$\eta_i(x, \lambda) \in (\min\{v(x), v_\lambda(x)\}, \max\{v(x), v_\lambda(x)\}), \qquad i = 1, 2.$$

We apply the "moving planes" method in three steps.

Step 1. There exists $\lambda^* > 0$ such that $U_\lambda \geqslant 0$ and $V_\lambda \geqslant 0$ in Σ_λ, for all $\lambda \geqslant \lambda^*$.

Let us prove the claim in Step 1 for U_λ. Assume for contradiction that for all $\lambda > 0$ there exists a point $x \in \Sigma_\lambda$ such that $U_\lambda(x) < 0$.

First, using (ii), we choose $\varepsilon_0 > 0$ such that $(\partial g/\partial u)(u, v) < 0$ and $(\partial f/\partial v)(u, v) < 0$ if $|u| + |v| < \varepsilon_0$, and then take $\bar{\lambda} > 0$ such that $u(x) + v(x) < \varepsilon_0$ when $|x| > \bar{\lambda}$. Next we observe that for all $\lambda > 0$ the function U_λ attains its infimum in Σ_λ, since it takes negative values in Σ_λ, is identically zero on $T_\lambda = \partial\Sigma_\lambda$, and tends to zero at infinity (note that $|x| \to \infty$ is equivalent to $|x^\lambda| \to \infty$, for fixed λ). We fix $\lambda \geqslant \bar{\lambda}$ and take $x_0 = x_0(\lambda) \in \Sigma_\lambda$ such that

$$U_\lambda(x_0) = \min_{x \in \Sigma_\lambda} U_\lambda(x) < 0,$$

so that $\Delta U_\lambda(x_0) \geqslant 0$. Then it follows from (3) that

$$\frac{\partial g}{\partial u}\,(\xi_1(x_0,\lambda), v(x_0))\; U_\lambda(x_0) \leqslant -\frac{\partial g}{\partial v}\,(u_\lambda(x_0), \eta_1(x_0,\lambda))\; V_\lambda(x_0). \tag{5}$$

Since $U_\lambda(x_0) < 0$ implies $\xi_1(x_0,\lambda) \leqslant u(x_0)$, we see that the left-hand side of (5) is strictly positive and deduce $V_\lambda(x_0) < 0$. Therefore we may take $x_1 = x_1(\lambda) \in \Sigma_\lambda$ such that

$$V_\lambda(x_1) = \min_{x \in \Sigma_\lambda} V_\lambda(x) < 0.$$

Now, using (4), we can repeat the above argument and show that $U_\lambda(x_1) < 0$ and

$$\frac{\partial f}{\partial v}\,(u(x_1), \eta_2(x_1,\lambda))\; V_\lambda(x_1) \leqslant -\frac{\partial f}{\partial u}\,(\xi_2(x_1,\lambda), v_\lambda(x_1))\; U_\lambda(x_1). \tag{6}$$

Let us put

$$\alpha(\lambda) = \frac{\partial g}{\partial u}\,(\xi_1(x_0,\lambda), v(x_0)) < 0, \qquad \beta(\lambda) = \frac{\partial g}{\partial v}\,(u_\lambda(x_0), \eta_1(x_0,\lambda)) \geqslant 0, \tag{7}$$

$$\gamma(\lambda) = \frac{\partial f}{\partial u}\,(\xi_2(x_1,\lambda), v_\lambda(x_1)) \geqslant 0, \qquad \delta(\lambda) = \frac{\partial f}{\partial v}\,(u(x_1), \eta_2(x_1,\lambda)) < 0. \tag{8}$$

By using (5) and (6) we obtain

$$\begin{aligned} U_\lambda(x_0) &\geqslant -\frac{\beta(\lambda)}{\alpha(\lambda)}\, V_\lambda(x_0) \\ &\geqslant -\frac{\beta(\lambda)}{\alpha(\lambda)}\, V_\lambda(x_1) \\ &\geqslant \frac{\beta(\lambda)\,\gamma(\lambda)}{\alpha(\lambda)\,\delta(\lambda)}\, U_\lambda(x_1) \\ &\geqslant \frac{\beta(\lambda)\,\gamma(\lambda)}{\alpha(\lambda)\,\delta(\lambda)}\, U_\lambda(x_0). \end{aligned}$$

The last quantity is strictly greater than $U_\lambda(x_0)$ provided that

$$a(\lambda) := \alpha(\lambda)\,\delta(\lambda) - \beta(\lambda)\,\gamma(\lambda) > 0.$$

Since U_λ and V_λ are both negative at x_0 and x_1, we have $u_\lambda(x_0) < u(x_0)$, $\xi_1(x_0, \lambda) \leqslant u(x_0)$, $\eta_1(x_0, \lambda) \leqslant v(x_0)$, $v_\lambda(x_1) < v(x_1)$, $\xi_2(x_1, \lambda) \leqslant u(x_1)$ and $\eta_2(x_1, \lambda) \leqslant v(x_1)$. The solutions decay at infinity so these inequalities imply

$$\lim_{\lambda \to \infty} a(\lambda) = \det A > 0,$$

which leads to a contradiction, for λ sufficiently large and greater than $\bar{\lambda}$. Step 1 is completed.

We put

$$\lambda_0 = \inf\{\lambda \in \mathbb{R} \mid U_\mu \geqslant 0 \text{ and } V_\mu \geqslant 0 \text{ in } \Sigma_\mu \text{ for all } \mu \geqslant \lambda\}.$$

Step 1 implies that $\lambda_0 < +\infty$. On the other hand $\lambda_0 = -\infty$ is impossible, as $U_\lambda(0) < 0$ for any $\lambda < -R$ with R chosen so that $\max_{|x_1| \geqslant R} u(x) < u(0)$. We conclude that λ_0 is finite.

Step 2. Either $U_{\lambda_0} \equiv 0$ or $V_{\lambda_0} \equiv 0$ in Σ_{λ_0}.

Since all objects we consider are continuous with respect to λ we already know that $U_{\lambda_0} \geqslant 0$ and $V_{\lambda_0} \geqslant 0$ in Σ_{λ_0}. Then it follows from (3), (4) and (i) that in Σ_{λ_0}

$$\Delta U_{\lambda_0} + \frac{\partial g}{\partial u}(\xi_1(x, \lambda_0), v(x))\, U_{\lambda_0} = -\frac{\partial g}{\partial v}(u_{\lambda_0}(x), \eta_1(x, \lambda_0))\, V_{\lambda_0} \leqslant 0 \tag{9}$$

$$\Delta V_{\lambda_0} + \frac{\partial f}{\partial v}(u(x), \eta_2(x, \lambda_0))\, V_{\lambda_0} = -\frac{\partial f}{\partial u}(\xi_2(x, \lambda_0), v_{\lambda_0}(x))\, U_{\lambda_0} \leqslant 0. \tag{10}$$

Applied to (9), the strong maximum principle implies that either $U_{\lambda_0} \equiv 0$ in Σ_{λ_0} or $U_{\lambda_0} > 0$ in Σ_{λ_0} with $\partial U_{\lambda_0}/\partial x_1 > 0$ on T_{λ_0}. By (10) the same holds for V_{λ_0}.

Therefore, we only have to exclude the situation when both U_{λ_0} and V_{λ_0} are strictly positive in Σ_{λ_0}, and have strictly positive x_1-derivatives on T_{λ_0}. Let us suppose this is the case.

The definition of λ_0 yields the existence of sequences $\{\lambda_k\}_{k=1}^{\infty} \subset \mathbb{R}$ and $\{x_k\}_{k=1}^{\infty} \subset \mathbb{R}^N$ such that $\lambda_k < \lambda_0$, $\lim_{k \to \infty} \lambda_k = \lambda_0$, $x_k \in \Sigma_{\lambda_k}$ and either U_{λ_k} or V_{λ_k} takes a negative value at x_k. Let for example $U_{\lambda_k}(x_k) < 0$ and rename x_k to be such that

$$U_{\lambda_k}(x_k) = \min_{x \in \Sigma_{\lambda_k}} U_{\lambda_k}(x) < 0.$$

Here we distinguish two cases.

Case 1. The sequence $\{x_k\}$ contains a bounded subsequence.

This case is treated in a standard way. We extract a subsequence of $\{x_k\}$ that converges to $x_0 \in \overline{\Sigma_{\lambda_0}}$. Since $U_{\lambda_0}(x_0) \leqslant 0$, necessarily $x_0 \in T_{\lambda_0}$. But x_k is an interior minimum of U_{λ_k}, so that $\nabla U_{\lambda_k}(x_k) = 0$, and thus $\nabla U_{\lambda_0}(x_0) = 0$. This contradicts $(\partial U_{\lambda_0}/\partial x_1)(x_0) > 0$ (see above).

Case 2. $|x_k| \to \infty$ as $k \to \infty$.

Ever since the first work on symmetry in $\mathbb{R}^N$ by Gidas, Ni, and Nirenberg [11] this case has been a basic issue in applying the "moving planes" method in unbounded domains. Fortunately, the machinery that we set up in Step 1 adapts to this case. Indeed, exactly as in Step 1 one can show that there exists an integer k_0 such that $V_{\lambda_k}(x_k) < 0$ when $k \geqslant k_0$ and that there exists $k_1 \geqslant k_0$ such that $U_{\lambda_k}(y_k)$ is negative when $k \geqslant k_1$, with y_k chosen so that

$$V_{\lambda_k}(y_k) = \min_{y \in \Sigma_{\lambda_k}} V_{\lambda_k}(y) < 0.$$

The corresponding inequalities (5) and (6) also hold and we finally obtain

$$\frac{\tilde{a}(k)}{\alpha(k)\,\delta(k)}\, U_{\lambda_k}(x_k) \geqslant 0,$$

where

$$\tilde{a}(k) = \alpha(k)\,\delta(k) - \beta(k)\,\gamma(k),$$

with $\alpha(k)$, $\beta(k)$, $\gamma(k)$, $\delta(k)$ defined analogously to (7) and (8)

$$\alpha(k) = \frac{\partial g}{\partial u}\,(\xi_1(x_k, \lambda_k), v(x_k)), \text{ etc.}$$

It is easy to see that

$$\lim_{k \to \infty} \tilde{a}(k) = \det A > 0,$$

and we obtain a contradiction for k sufficiently large. This argument completes Step 2.

Step 3. Conclusion.

Let for example $U_{\lambda_0} \equiv 0$. Then $U_\lambda \geqslant 0$ in Σ_λ for all $\lambda > \lambda_0$ and it is straightforward to see that

$$\operatorname{sign}(x_1 - \lambda_0)\,\frac{\partial u}{\partial x_1}\,(x_1, x') \leqslant 0 \tag{11}$$

for all $x=(x_1, x')\in\mathbb{R}\times\mathbb{R}^{N-1}$. Next, we observe that the function v satisfies the single equation

$$\Delta v+\bar{f}(x, v)=0,$$

with $\bar{f}(x, v)=f(u(x), v)$. From our hypotheses it is clear that $(\partial\bar{f}/\partial v)(x, v)$ is negative for small v and large $|x_1|$. In view of (i) and (11) we have

$$\operatorname{sign}(x_1-\lambda_0)\frac{\partial\bar{f}}{\partial x_1}(x_1, x', v)=\operatorname{sign}(x_1-\lambda_0)\frac{\partial f}{\partial u}(u(x), v)\frac{\partial u}{\partial x_1}(x_1, x')\leqslant 0,$$

for all $x\in\mathbb{R}^N$, $v\in\mathbb{R}$. This is exactly what we need in order to apply the results for single equations (see [13]) and conclude that there exists some λ_0' with $\lambda_0'\leqslant\lambda_0$ such that v is symmetric with respect to $T_{\lambda_0'}$. Alternatively, to prove this one could use the reasonings in Steps 1 and 2, combined with moving planes coming from $-\infty$.

Finally, since $U_\lambda>0$ and $V_\lambda>0$ in Σ_λ for $\lambda>\lambda_0$, by using (i) we see that Hopf's lemma, applied to (3), yields

$$\frac{\partial U_\lambda}{\partial x_1}(\lambda, x')>0\qquad\text{for all}\quad\lambda>\lambda_0,\qquad x'\in\mathbb{R}^{N-1}.$$

Analogously, since $U_\lambda<0$ and $V_\lambda<0$ in Σ_λ for $\lambda<\lambda_0'$, we infer from (4)

$$\frac{\partial V_\lambda}{\partial x_1}(\lambda, x')<0\qquad\text{for all}\quad\lambda<\lambda_0',\qquad x'\in\mathbb{R}^{N-1}.$$

Since

$$\frac{\partial U_\lambda}{\partial x_1}(\lambda, x')=-2\frac{\partial u}{\partial x_1}(\lambda, x')\qquad\text{and}\qquad\frac{\partial V_\lambda}{\partial x_1}(\lambda, x')=-2\frac{\partial v}{\partial x_1}(\lambda, x'),$$

the proof of Theorem 2 is complete.

2.2. *The General Case*

The purpose of this section is to prove Theorem 2. In order to show that all u_i are radially symmetric with respect to the same origin, it is enough to establish that, given an arbitrary direction $\gamma\in\mathbb{R}^N\setminus\{0\}$, there exists $\lambda=\lambda(\gamma)$ such that all u_i are symmetric with respect to the hyperplane $T_\lambda=\{x\in\mathbb{R}^N\mid x\cdot\gamma=\lambda\}$. Indeed, it is then easy to see that $\bigcap_{i=1}^n T_{\lambda(e_i)}$ is a point of symmetry.

In the sequel, we fix a direction γ. We denote by $x\to x^\lambda$ the reflection with respect to T_λ, and by U_i^λ, $i=1, ..., n$, the difference functions

$$U_i^\lambda(x)=u_i(x^\lambda)-u_i(x),$$

defined in $\Sigma_\lambda = \{x \in \mathbb{R}^N \mid x \cdot \gamma > \lambda\}$. As in Section 2.1, the proof is carried out in three steps. In the first step we show that

$$\Lambda = \inf\{\lambda > 0 \mid U_i^\mu \geqslant 0 \text{ in } \Sigma_\mu \text{ for } i = 1, ..., n, \text{ and all } \mu \geqslant \lambda\}$$

is well-defined, i.e. $\Lambda < +\infty$. The second step consists in proving that either

(a) $\Lambda = 0$, or

(b) $\Lambda > 0$ and $U_i^\Lambda \equiv 0$ for all $i = 1, ..., n$.

The symmetry conclusion then follows easily (Step 3).

Step 1. $\Lambda < +\infty$.

Since all u_i tend to zero at infinity, we can fix some large $R_0 \geqslant R_1$ such that $|u| < \varepsilon$, in $\mathbb{R}^N \backslash B_{R_0}$ (ε and R_1 are defined in (H3), (H4)). We take $\lambda^* > R_0$, for which

$$\max_{\substack{1 \leqslant i \leqslant n \\ x \in \bar{B}^\lambda_{R_0}}} u_i(x) < \min_{\substack{1 \leqslant i \leqslant n \\ x \in \bar{B}_{R_0}}} u_i(x),$$

for all $\lambda > \lambda^*$, where $\bar{B}^\lambda_{R_0} = \{x \mid x^\lambda \in \bar{B}_{R_0}\}$. Hence $U_i^\lambda > 0$ in $\bar{B}^\lambda_{R_0} \subset \Sigma_\lambda$ for all $\lambda > \lambda^*$. We shall show that $U_i^\lambda > 0$ in the remaining part $\Sigma_\lambda \backslash \bar{B}^\lambda_{R_0}$. For this purpose, writing equations (1) at x and x^λ, and using Taylor's expansion, we notice that the functions U_i^λ satisfy the following system of linear partial differential equations

$$\Delta U_i^\lambda + \frac{\partial f}{\partial r}(\eta)(r^\lambda - r) + \sum_{1 \leqslant j \leqslant n} \frac{\partial f_i}{\partial u_j}(r, \xi_{i1}, ..., \xi_{in})\, U_j^\lambda = 0, \qquad i = 1, ..., n, \tag{12}$$

where $\eta = \eta(x, \lambda) \in \mathbb{R}^{n+1}_+$ and

$$\xi_{ij} = \xi_{ij}(x, \lambda) \in (\min(u_j(x), u_j(x^\lambda)), \max(u_j(x), u_j(x^\lambda))).$$

We have $|x^\lambda| = r^\lambda < r = |x|$ for $x \in \Sigma_\lambda$, and we obtain from (H1) the following systems of inequalities for U_i^λ

$$\Delta U_i^\lambda + \sum_{1 \leqslant j \leqslant n} \frac{\partial f_i}{\partial u_j}(r, \xi_{i1}, ..., \xi_{in})\, U_j^\lambda \leqslant 0, \qquad i = 1, ..., n. \tag{13}$$

In order to realize assumption (H4) with strictly positive n-principal minors, we shift the diagonal coefficients of the matrix A by making the following change of functions

$$\bar{U}_i^\lambda = \frac{U_i^\lambda}{g},$$

where

$$g(x) = \begin{cases} |x|^{-(N-2)/2} + 1 & \text{if} \quad N \geqslant 3 \\ \ln(\ln(|x| + 27)) & \text{if} \quad N = 2. \end{cases}$$

Simple calculations yield $g \geqslant 1$ and $\Delta g < 0$ in $\mathbb{R}^N \backslash \{0\}$. Such a transform is classical in the scalar case, see [13]. See also [2] for special systems in two dimensions.

It is easy to see that the new functions satisfy the following system:

$$\Delta \bar{U}_i^\lambda + 2 \frac{\nabla g}{g} \nabla \bar{U}_i^\lambda + \sum_{1 \leqslant j \leqslant n} \left(\frac{\partial f_i}{\partial u_j} (r, \xi_{i1}, ..., \xi_{in}) + \delta_{ij} \frac{\Delta g}{g} \right) \bar{U}_j^\lambda \leqslant 0, \tag{14}$$

for $i = 1, ..., n$. We want to show that $\bar{U}_i^\lambda \geqslant 0$ (and thus $U_i^\lambda \geqslant 0$) in Σ_λ, for all $\lambda > \lambda^*$. We argue by contradiction. Suppose there exist $\lambda > \lambda^*$ and $i_0 \in \{1, ..., n\}$, for which $\inf_{\Sigma_\lambda} \bar{U}_{i_0}^\lambda < 0$. We set $J = \{ j \mid \bar{U}_j^\lambda \geqslant 0 \text{ in } \Sigma_\lambda \} \subsetneqq \{1, ..., n\}$ (J may be empty), and $I = \{1, ..., n\} \backslash J$ (note that $i_0 \in I$). We consider only those inequalities in (14) which correspond to indices $i \in I$. Since $\bar{U}_j^\lambda \geqslant 0$ in Σ_λ for $j \in J$, by (H2) they still hold if one cancels all terms containing $\bar{U}_j^\lambda$ for $j \in J$. What we get, up to a permutation of the indices, is a set of inequalities of type (14) for $i = 1, ..., p$, with $p = |I|$. We note that the permutation does not affect assumptions (H2), (H4) and that they remain valid for the submatrix $(\partial f_i / \partial u_j)_{1 \leqslant i, j \leqslant p}$. Indeed, this is trivial for (H2), while for (H4) it follows from Lemma 2.2 in [6], the statement of which we give here for the reader's convenience.

LEMMA 6. *Let $M = (m_{ij})_{1 \leqslant i, j \leqslant n}$ be a matrix such that $m_{ij} \leqslant 0$ for $i \neq j$. Assume all n-principal minors of M have positive determinants. Then*

(i) *all minors of M obtained by dropping lines and columns of the same order have positive determinants*;

(ii) *if M_{ij} is the minor of M obtained by dropping the ith line and the jth column we have*

$$(-1)^{i+j} \det M_{ij} \geqslant 0.$$

Since $\inf_{\Sigma_\lambda} \bar{U}_i^\lambda < 0$ for all $i = 1, ..., p$, and $\bar{U}_i^\lambda > 0$ in $\bar{B}_{R_0}^\lambda$, $\bar{U}_i^\lambda \to 0$ at infinity (here we use $g \geqslant 1$), we may take $x_1, ..., x_p \in \Sigma_\lambda \setminus \bar{B}_{R_0}^\lambda$ such that $\bar{U}_i^\lambda(x_i) = \min_{\Sigma_\lambda} \bar{U}_i^\lambda < 0$ (which implies $\Delta \bar{U}_i^\lambda(x_i) \geqslant 0$ and $\nabla \bar{U}_i^\lambda(x_i) = 0$). Writing the equations in (14) respectively at $x_1, ..., x_p$ and using the fact that $\bar{U}_j^\lambda(x_j) \leqslant \bar{U}_j^\lambda(x_i)$ results in

$$\sum_{1 \leqslant j \leqslant p} \left(\frac{\partial f_i}{\partial u_j} (r, \xi_{i1}, ..., \xi_{ip}) + \delta_{ij} \frac{\Delta g}{g} \right) \bar{U}_j^\lambda(x_j) \leqslant 0, \qquad i = 1, ..., p. \tag{15}$$

This can be written in terms of matrices as

$$M\bar{U} = Y \tag{16}$$

where $Y = (y_1, ..., y_p)$, $M = (m_{ij})_{1 \leqslant i,\, j \leqslant p}$, with

$$y_i \geqslant 0, \; m_{ij} = -\left(\frac{\partial f_i}{\partial u_j} (r, \xi_{i1}, ..., \xi_{ip}) + \delta_{ij} \frac{\Delta g}{g} \right), \qquad i, j = 1, ..., p,$$

and $\bar{U} = (\bar{U}_1^\lambda(x_1), ..., \bar{U}_p^\lambda(x_p))$. Since $x_1, ..., x_p \in \Sigma_\lambda \setminus \bar{B}_{R_0}^\lambda$, we have $r_i > R_0$, so using the choice of λ^* as in Section 2.1 we can see that $\xi_{ij}(x_k) \in (0, \varepsilon)$. Besides, we know that $\Delta g/g < 0$. Assumptions (H2) and (H4) therefore yield $m_{ij} \leqslant 0$ for $i \neq j$, and all n-principal minors of M have positive determinants.

Since M is invertible ($\det M > 0$), relation (16) yields $\bar{U} = M^{-1} Y$. Since $y_i \geqslant 0$, $i = 1, ..., p$, it follows from Cramer's formula and statement (ii) of the lemma above that $\bar{U}_i^\lambda(x_i) \geqslant 0$, $i = 1, ..., p$. But we have taken x_i to be such that $\bar{U}_i^\lambda(x_i) < 0$ — a contradiction.

Hence $\Lambda \leqslant \lambda^* < +\infty$.

Step 2. Either $\Lambda = 0$, or $\Lambda > 0$ and $\bar{U}_i^\Lambda \equiv 0$, $i = 1, ..., n$.

We argue by contradiction. Suppose $\Lambda > 0$ and $\bar{U}_{i_0}^\Lambda \not\equiv 0$ for some $i_0 \in \{1, ..., n\}$. By the definition of Λ, we see that $U_i^\Lambda \geqslant 0$ for all $i = 1, ..., n$. Hence the strong maximum principle, applied to each equation in (14), implies that either $U_i^\Lambda > 0$ or $U_i^\Lambda \equiv 0$ in Σ_Λ. Now $U_{i_0}^\Lambda > 0$, and the coupling condition (H3) implies that $U_i^\Lambda > 0$ in Σ_Λ for all $i = 1, ..., n$. Indeed, by (H3), there exists $j_0 \in \{1, ..., n\} \setminus \{i_0\}$ such that

$$\frac{\partial f_{j_0}}{\partial u_{i_0}} > 0 \qquad \text{in} \quad \mathcal{O}. \tag{17}$$

If $U_{j_0}^\Lambda \equiv 0$, by writing the j_0th inequality in (14), we get

$$\sum_{j \neq j_0} \frac{\partial f_{j_0}}{\partial u_j} (r, \xi_{j_0 1}, ..., \xi_{j_0 n}) \, U_j^\Lambda \leqslant 0,$$

which contradicts $U_{i_0}^{\Lambda} > 0$ for $|x| > R_0$, $x \notin \bar{B}_{R_0}^{\Lambda}$, in view of (H2) and (17). Hence $U_{j_0}^{\Lambda} > 0$. By (H3) we can choose $k_0 \in \{1, ..., n\} \setminus \{i_0, j_0\}$ such that either $\partial f_{i_0}/\partial u_{k_0}$ or $\partial f_{j_0}/\partial u_{k_0}$ is positive in $\mathcal{O}$. As above this means that $U_{k_0}^{\Lambda} > 0$, so, repeating the same argument n times, we conclude that we have $U_i^{\Lambda} > 0$ and $\bar{U}_i^{\Lambda} > 0$ in Σ_{Λ}, for all $i = 1, ..., n$. By the definition of Λ we get a sequence $\lambda_k \nearrow \Lambda$ such that $\min_{1 \leqslant i \leqslant n} \inf_{\Sigma_{\lambda_k}} \bar{U}_i^{\lambda_k} < 0$. By the argument in Step 1, and up to an extraction of a subsequence, we can construct $x_k \in \Sigma_{\lambda_k}$, such that

$$\bar{U}_{i_0}^{\lambda_k}(x_k) = \min_{\Sigma_{\lambda_k}} \bar{U}_{i_0}^{\lambda_k} < 0$$

for some $i_0 \in \{1, ..., n\}$. Therefore, up to a further subsequence, there are two cases to consider.

Case 1. $x_k \to \bar{x}$.

Since $\bar{U}_{i_0}^{\Lambda} > 0$ in Σ_{Λ} and $\bar{U}_{i_0}^{\Lambda}(\bar{x}) \leqslant 0$, necessarily $\bar{x} \in T_{\Lambda}$, and $\bar{U}_{i_0}^{\Lambda}(\bar{x}) = 0$. Furthermore $\nabla \bar{U}_{i_0}^{\Lambda}(\bar{x}) = 0$.

Since $\bar{U}_i^{\Lambda} > 0$ in Σ_{Λ}, $i = 1, ..., n$, the i_0th equation in (14) yields

$$\Delta \bar{U}_{i_0}^{\Lambda} + 2\, \frac{\nabla g}{g}\, \nabla \bar{U}_{i_0}^{\Lambda} + \left(\frac{\partial f_{i_0}}{\partial u_{i_0}} + \frac{\Delta g}{g} \right) \bar{U}_{i_0}^{\Lambda} \leqslant 0 \text{ in } \Sigma_{\Lambda},$$

giving a contradiction (Hopf's Lemma).

Case 2. $|x_k| \to +\infty$.

Then, for sufficiently large k, we have $x_k \in \Sigma_{\lambda_k} \setminus \bar{B}_{R_0}^{\lambda_k}$, and the same argument as in Step 1 provides a contradiction.

Step 3. Conclusion.

We have reached the following result: either $\Lambda = 0$, or $\Lambda > 0$ and $U_i^{\Lambda} \equiv 0$ in Σ_{Λ}, $i = 1, ..., n$. Now the conclusion follows easily. If $\Lambda > 0$, we are done. If $\Lambda = 0$, we have $U_i^0 \geqslant 0$ in Σ_0 and, repeating steps 1 and 2 in the opposite direction $(-\gamma)$, we get that either there exists $\Lambda' > 0$ such that $U_i^{\Lambda'} \equiv 0$ or $U_i^0 \leqslant 0$, $i = 1, ..., n$, so that in each case there is some λ_0 for which the u_i are symmetric with respect to the hyperplane T_{λ_0}. It is then standard to infer that all u_i are radially symmetric with respect to some point $x_0 \in \mathbb{R}^N$. It is easy to prove, as in Section 2.1, that $du_i/dr < 0$, for $r = |x - x_0| > 0$.

The proof of Theorem 2 is complete.

3. DISCUSSION

For simplicity in this section we consider the model case of two equations that we already described in Section 2.1.

We start by pointing out that Theorem 2 fails if we consider non-autonomous systems or systems of three or more equations. A counter-example is provided by the system

$$\begin{cases} \Delta u - u + u^p = 0 \\ \Delta v - v + v^p = 0 \\ \Delta w - w + u + v^2 = 0 \end{cases}$$

where $1 < p < (N+2)/(N-2)$. Taking $u = u(|x|)$ to be the unique positive (exponentially decreasing) solution of the first equation and setting $v = u(|x - x_0|)$, with $x_0 \neq 0$, we see that w cannot be symmetric.

Next, we are going to show that if all hypotheses (i)–(iii) are satisfied, except one of (ii) and (iii), in which the inverse inequality is strict, then no positive solution of (2) can exist.

Let u and v be solutions of (2) and let us put

$$\alpha = \frac{\partial g}{\partial u}(0, 0), \qquad \beta = \frac{\partial g}{\partial v}(0, 0)$$

$$\gamma = \frac{\partial f}{\partial u}(0, 0), \qquad \delta = \frac{\partial f}{\partial v}(0, 0).$$

We suppose first that $\alpha > 0$. We distinguish two cases.

Case 1. $\beta > 0$.

In this case Taylor's expansion yields

$$\Delta u + \alpha u + \beta v + o(u + v) = 0,$$

where $o(t)$ is a quantity such that $o(t)/t \to 0$ as $t \to 0$. Without loss of generality we have supposed $f(0, 0) = g(0, 0) = 0$, otherwise (2) has no solutions.

We fix $\varepsilon_0 > 0$ such that

$$|o(t)| \leqslant \tfrac{1}{2} \min\{\alpha, \beta\}\ |t|$$

if $|t| \leqslant \varepsilon_0$, and take $R_1 > 0$ such that $u(x) + v(x) < \varepsilon_0$ if $|x| \geqslant R_1$. We obtain that, in $\mathbb{R}^N \backslash B_{R_1}$,

$$\Delta u + \tfrac{1}{2}\alpha u + \tfrac{1}{2}\beta v \leqslant 0.$$

Hence

$$\Delta u + \tfrac{1}{2}\alpha u \leqslant 0 \qquad \text{in } \mathbb{R}^N \backslash B_{R_1}.$$

Then a well-known sufficient condition for the maximum principle (see [14], for instance) implies that it holds for the operator $\Delta+\frac{1}{2}\alpha$ in $C(R_2)=\{x\in\mathbb{R}^N : R_1<|x|<R_2\}$, for any $R_2>R_1$. This leads to a contradiction for R_2 sufficiently large, for example, taken such that

$$\lambda_1(-\Delta, C(R_2))<\frac{\alpha}{4}.$$

Case 2. $\beta=0$.

In this case (iii) implies $\delta>0$. By Taylor's expansion and (i) we obtain

$$\begin{cases}\Delta u+\alpha u+o(u+v)=0\\ \Delta v+\delta v+o(u+v)\leqslant 0.\end{cases}$$

Hence, by addition,

$$\Delta(u+v)+\min\{\alpha, \delta\}(u+v)+o(u+v)\leqslant 0,$$

and we conclude as in case 1.

Suppose next that $\det A<0$. In this case (i) and (ii) imply that $\beta>0$ and $\gamma>0$. Again by Taylor's expansion we obtain

$$\begin{cases}\Delta u+\alpha u+\beta v+o(u+v)=0\\ \Delta v+\gamma u+\delta v+o(u+v)=0\end{cases}$$

and consequently

$$\begin{cases}\Delta u+(\alpha-\varepsilon)\, u+(\beta-\varepsilon)\, v\leqslant 0\\ \Delta v+(\gamma-\varepsilon)\, u+(\delta-\varepsilon)\, v\leqslant 0\end{cases} \tag{18}$$

in $\mathbb{R}^N\backslash B_{R_1}$, for some large ball B_{R_1}, where $\varepsilon>0$ is chosen such that the "perturbed" matrix A_ε still has negative determinant.

A sufficient condition for the maximum principle to hold for a linear system like (18) was derived in [7]. It implies that the maximum principle holds for the operator $\vec{\Delta}+A_\varepsilon$ in $C(R_2)$, for all $R_2>R_1$. On the other hand, it was proved in [6] that a necessary condition for the maximum principle to hold in $C(R_2)$ is

$$\det[\lambda_1(-\Delta, C(R_2))\, Id-A_\varepsilon]>0,$$

which leads to a contradiction if R_2 is chosen so that

$$[\lambda_1(-\Delta, C(R_2))]^2+\lambda_1(-\Delta, C(R_2))\operatorname{tr} A_\varepsilon<-\det A_\varepsilon.$$

Let us also remark that the case when one of the quantities in (ii) and (iii) is equal to zero and (H4) does not hold appears to be quite difficult. Of course in that case there can be existence and by now only very partial symmetry results are available (see [1], [3], [18] for scalar equations and [2], [9], for systems). Finally, we note that in last years there have been some results which established maximum principles for non-cooperative systems. We do not know if a symmetry result can be proved in this case. We intend to investigate this question in the future.

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